On Existence and Locally Attractivity Results for Fractional Order Nonlinear Random Integral Equation

B. D. Karande[1], K. D. Jagtap[2] & S. V. Badgire[3]
[1]Department of Mathematics, Maharashtra Udayagiri College, Udgir, Maharashtra, India
[2]Department of Mathematics, Dayanand Science College, Latur, Maharashtra, India
[3]Department of Mathematics, Azad Mahavidyalaya, Ausa, Maharashtra, India

Abstract

In this paper, we examine the existence and qualitative behaviour of solutions for a class of nonlinear random integral equations of fractional order in $R_+ = [0, \infty)$. The analysis is carried out within the framework of Banach algebra, employing a hybrid fixed-point theorem as the principal tool. The problem is considered under the assumptions of Lipschitz continuity and Caratheodory conditions, which ensures the measurability and continuity properties required for the existence of random solutions. In addition to proving the existence of such solutions, we establish their local attractivity, thereby demonstrating the stability of the system in a probabilistic sense. Moreover, we have proved the existence of extremal solutions. The theoretical results presented in this work contribute to the growing field of fractional calculus and stochastic analysis by providing a rigorous framework for studying fractional random integral equations. To illustrate the applicability of the main results, we provided a concrete example that verifies the theoretical findings and highlights the practical relevance of the proposed approach.

Keyword: Fixed point theorems, Nonlinear integral equations, Random operators, Initial value problems.

Mathematics subject classification: 47H10, 45G10, 60H25, 34A12.

1. Introduction

Differentiations and integrations of any non-integer order are generalized in fractional calculus. In many engineering and scientific fields, including physics, chemistry, electrodynamics, aerodynamics, economics, and more, fractional calculus is a concept that appears in the mathematical modeling of systems and processes (Akgul and Khoshnaw 2020; Hammachukiattikul and Mohanapriya 2020; Boulaaras and Jan 2023; Cheow and Ng 2024). The fixed-point theorem has recently attracted a lot of writers to fractional differential equations, and numerous findings about linear and nonlinear fractional differential equations have been published in the literature (Damag and Kilicman 2020; Abbas and Arifi 2019; Zhang and Hu 2019; Liu and Liu 2024). The innovative properties of attractivity and asymptotic

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attractivity of solutions have been utilized in the relatively new research of nonlinear integral equations with unbounded intervals. These characteristics of solutions can be handled in two ways. classical fixed-point theorems that use topological and analytical hypotheses and fixed-point theorems that use non-compactness measures. The existence and uniqueness theorems of nonlinear random equations can be demonstrated using random fixed-point theorems, which are stochastic generalizations of classical or deterministic fixed-point theorems in abstract spaces. For detail discussion of fixed-point theorem in probabilistic analysis refer (Bharucha-Ried 1976).

The theory of fractional order integral equations has attracted a lot of attention and is prominent part of nonlinear analysis. There have been numerous research papers and monographs published on integer order and fractional order differential and integral equations (Arunchai and Plubtieng 2013; Dhage and Ntouyas 2010; Damag and Kilicman 2020; Granas and Dugundji 2003; Podlubny 1993; Machado and Silva 2010; Abbas and Arifi 2019; Abbas and Benchohra 2019; Zhang and Hu 2019). Among these research paper some discuss various form of existence results and some discuss numerical methods of solutions for fractional and integer order equations. In the present paper, we have discussed the existence of solutions and locally attractivity results of the nonlinear random integral equation of fractional order in the right half R_+ of the real line R. Moreover, we have discussed the existence of extremal solutions.

2. Preliminaries

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in paper.

2.1 Definition: (Arunchai and Plubtieng 2013; Dhage and Ntouyas 2010) A mapping $\mathscr{f}: \Omega \to \mathfrak{B}$, where $\mathfrak{B}$ be a separable Banach space with $\beta_{\mathscr{f}}$ σ-algebra of all Borel subsets of $\mathfrak{B}$ and (Ω, A) be a measurable space is called measurable if for any Borel subset B of $\mathfrak{B}$,

$$\mathscr{f}^{-1}(B) = \{\upsilon \in \Omega : \mathscr{f}(\upsilon) \in B\} \in A$$

2.2 Definition: (Bharucha-Ried 1976; Arunchai and Plubtieng 2013; Granas and Dugundji 2003) A mapping $f: \Omega \times \mathbb{A} \to \mathbb{A}$ is known as random operator if $\upsilon \to f(\upsilon, a)$ is measurable for each $a \in \mathbb{A}$ and this random operator is generally denoted as $f(\upsilon)a = f(\upsilon, a)$.

2.3 Definition: (Bharucha-Ried 1976; Arunchai and Plubtieng 2013) A mapping $g: \Omega \times \mathbb{B} \to \mathbb{B}$ random operator, a random variable $\varsigma: \Omega \to \mathbb{B}$ is called random fixed point of a random operator, if $g(\upsilon)\varsigma(\upsilon) = \varsigma(\upsilon)$ for every $\upsilon \in \Omega$.

2.4 Definition: (Arunchai and Plubtieng 2013) A random operator $f(\upsilon)$ is totally bounded and continuous on $\mathbb{A}$ then f is called completely continuous on Banach space $\mathbb{A}$.

2.5 Definition: (Hammachukiattikul and Mohanapriya 2020) A class of measurable functions $\{y_n(\upsilon)\}$ is said to be equicontinuous class, if for every $\varepsilon > 0$ there exist$\delta > 0$ such that $\|y_n(t_1, \upsilon) - y_n(t_2, \upsilon)\| < \varepsilon$ whenever $|t_1 - t_2| < \delta$ for all $n = 1, 2, 3, \ldots$

2.6 Theorem (Arzela-Ascoli Theorem): (Hammachukiattikul and Mohanapriya 2020**)** If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of functions in $C(R_+, R)$, then it has a convergent subsequence.

2.7 Theorem: (Malik and Arora 1982) A metric space X is compact if and only if every sequence in X has a convergent subsequence.

2.8 Definition: (Dhage and Ntouyas 2010) A random operator $g: \Omega \times \mathbb{B} \to \mathbb{B}$ is called a $\mathfrak{D}$-Lipschitzician, if there exist a nondecreasing continuous function $\varphi: \Omega \times R_+ \to R_+$ such that for every $\upsilon \in \Omega$

$$\|g(\upsilon)a - g(\upsilon)b\| \leq \varphi_\upsilon(\|a - b\|)$$

For all $a, b \in \mathbb{B}$, where $\varphi_\upsilon(r) = \varphi(\upsilon, r)$ with $\varphi(\upsilon, 0) = 0$.

Here some special case $\varphi_\upsilon(r) = \beta(\upsilon)r$, for $\beta(\upsilon) > 0$ for every $\upsilon \in \Omega$ then the random operator $g(\upsilon)$ is called Lipschitz with Lipschitz constant $\beta(\upsilon)$, for $\upsilon \in \Omega$. In particular for each $\upsilon \in \Omega$, $\beta(\upsilon) < 1$ then random operator $g(\upsilon)$ is known as a contraction with contraction constant $\beta(\upsilon)$.

2.9 Definition: (Dhage and Ntouyas 2010; Abbas and Benchohra 2014) Random solutions of random equations are said to be locally attractive on R_+. If there is a closed ball $\bar{B}_a(x_0)$ in the space $S = BC(R_+, R)$and for some $x_0 \in S$ and for some real number$a > 0$, such that for arbitrary random solutions $x = x(t, \upsilon)$ and $y = y(t, \upsilon)$ of the random equation belonging to $\bar{B}_a(x_0) \cap \mathfrak{H}$, where $\mathfrak{H}$ is a non-empty subset of S, we have

$$\lim_{t\to\infty} |x(t, \upsilon) - y(t, \upsilon)| = 0 \text{ for all } \upsilon \in \Omega$$

2.10 Theorem: (Dhage 2003) Let C be a closed and bounded subset of a separable Banach algebra $\mathfrak{B}$ and let $\mathcal{S}(\upsilon), \mathcal{T}(\upsilon) : \Omega \times C \to \mathfrak{B}$ be two random operators satisfying for each $\upsilon \in \Omega$

- $\mathcal{S}(\upsilon)$ is $\mathfrak{D}$-Lipschitz with $\mathfrak{D}$-function φ
- $\mathcal{T}(\upsilon)$ is completely continuous, and
- $\mathcal{S}(\upsilon)y\, \mathcal{T}(\upsilon)y \in C$ for each $y \in C$.
  Then the random equation $\mathcal{S}(\upsilon)y\mathcal{T}(\upsilon)y = y$ has a random solution whenever $\mathcal{M}(\upsilon)\varphi_\upsilon(r) < r, r > 0$, for each $\upsilon \in \Omega$ where $\mathcal{M}(\upsilon) = \|\mathcal{T}(\upsilon)(C)\|$.

2.11 Corollary: (Dhage 2003) Let C be a closed, convex and bounded subset of a separable Banach algebra $\mathfrak{B}$ and let

$\mathcal{S}(\upsilon), \mathcal{T}(\upsilon) : \Omega \times C \to \mathfrak{B}$ be two random operators satisfying for each $\upsilon \in \Omega$,

- $\mathcal{S}(\upsilon)$ is Lipschitz with Lipschitz constant $\beta(\upsilon)$
- $\mathcal{T}(\upsilon)$ is continuous and compact,
- $\mathcal{S}(\upsilon)y\, \mathcal{T}(\upsilon)y \in C$ for each $y \in C$.
  Then the random equation

$$\mathcal{S}(\upsilon)y\, \mathcal{T}(\upsilon)y = y$$

has a random solution and the set of such solutions is compact whenever $\beta(v)\mathcal{M}(v) < 1$ for each $v \in \Omega$, where $\mathcal{M}(v) = \|\mathcal{T}(v)(C)\|$.

3. Results and Discussion

Let $\xi, \zeta \in (0,1)$ and R denote the real numbers where as R_+ be the set of non-negative numbers i.e.

$R_+ = [0, \infty) \subset R$.. Consider the fractional order nonlinear random integral equation (FONRIE)

$$y(t,v) = \left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\, p(s, y(s,v), v)\ ds\right]$$

$$\times \left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, y(s,v), v) ds\right] \tag{1}$$

For all $t \in R_+, v \in \Omega$

Where $p: R_+ \times R \times \Omega \to R - \{0\}$; $g: R_+ \times \Omega \to R$; $q: R_+ \times R \times \Omega \to R$ and by solution of equation (1) we mean a function $y \in BM(R_+, R)$ that satisfies (1) on R_+, Where $BM(R_+, R)$ is space of bounded measurable real valued functions defined on R_+. Define a supremum norm $\|.\|$ in $BM(R_+, R)$ by $\|y\| = \sup_{t \in R_+} |y(t)|$. Clearly $BM(R_+, R)$ is a separable Banach algebra with this maximum norm.

We seek random solutions of FONRIE (1) in the space $BM(R_+, R)$ of bounded measurable and real valued function on R_+ . L^1- norm in $L^1(R_+, R)$ is defined by

$$\|y\|_{L^1} = \int_0^t |y(t)|\ ds\,.$$

3.1 Definition: (Arunchai and Plubtieng 2013; Dhage and Ntouyas 2010) A mapping $\lambda: R_+ \times R \times \Omega \to R$ is said to satisfy condition of $L(v)$-caratheodory if

i) The map $(t, v) \to \lambda(t, y, v)$ is measurable for all $y \in R$ and

ii) The map $y \to \lambda(t, y, v)$ is continuous for all $t \in R_+$, $v \in \Omega$

Furthermore $L^1(v)$-caratheodory if

iii) There exits measurable and bounded function $\psi: \Omega \to L^1(R_+)$ such that $|\lambda(t, y, v)| \le \psi(t, v)$ a.e. $t \in R_+$ for all $v \in \Omega$ and $y \in R$.

We consider following hypothesis

(H_0) The function $v \to p(t, y, v)$ is measurable for all $t \in R_+$, $y \in R$.

(H_1) The function $t \to p(t, y, v)$ is Riemann integrable for each $y \in R, v \in \Omega$.

(H_2) There exist a function $\mathcal{L}: \Omega \to L^1(R_+, R)$ such that for each $v \in \Omega$

$$|p(t, y(t,v), v) - p(t, z(t,v), v)| \le \mathcal{L}(t,v)|y(t,v) - z(t,v)| \quad a.e.$$

$t \in R_+,\ \upsilon \in \Omega,\ \ y, z \in R$, moreover $K_2 = \sup\{\frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\, \mathcal{L}(s,\upsilon)\, ds\,\}$

(H3) There exist a function $\phi\colon \Omega \to L^1(\,R_+, R)$ such that for each $\upsilon \in \Omega$

$|p(t,y,\upsilon)| \le \phi(t,\upsilon);\ \ a.e.$ for all $t \in R_+,\ y \in R$.

(H4) The function $g\colon R_+ \times \Omega \to R$ be measurable bounded and vanish at $t \longrightarrow \infty$ i.e.

$\lim_{t\to\infty} g(t,\upsilon) = 0$

(H5) The function $(t,\upsilon) \to q(t,y,\upsilon)$ is measurable for all $y \in R$.

(H6) The function q is $L^1(\upsilon)$-caratheodory and there exist function

$$\psi_r\colon \Omega \to L^1(R_+, R)\ a.e.\, t \in R_+$$

such that $|q(t, y(t,\upsilon), \upsilon)| \le \psi_r(t,\upsilon)$ for all $(t,y) \in R_+ \times R$

Remark: Note that the hypothesis H5, H6 hold then there exist continuous bounded function

$\mathcal{V}\colon R_+ \times \Omega \to R_+$ defined by $\mathcal{V}(t,\upsilon) = \int_0^t (t-s)^{\zeta-1}\, \psi_r(s,\upsilon) ds$ with

$K_1 = \sup\left\{\frac{1}{\Gamma(\zeta)}\mathcal{V}(t,\upsilon) : t \in R_+, \upsilon \in \Omega\right\}$ and vanish at $t = \infty$.

3.2 Theorem: If the hypothesis H0 – H6 hold and $K_2(\|g(\upsilon)\| + K_1) < 1$ then the FONRIE (1) has a random solution in R_+ and moreover, the random solutions are locally attractive on R_+.

Proof: Let $\mathfrak{B} = BM(R_+, R)$ be a measurable Banach algebra and define a subset C of $\mathfrak{B}$ as

$$C = \{y \in \mathfrak{B} : \|y\| \le r\} = B_r[0]$$

Where, $r = \phi_1(\|g(\upsilon)\| + K_1)$

Now we define two operators $\mathcal{S}\colon \Omega \times C \to \mathfrak{B};\ \ \mathcal{T}\colon \Omega \times C \to \mathfrak{B}$;

$$\mathcal{S}(\upsilon)y(t) = \frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{(\xi-1)} p(s, y(s,\upsilon), \upsilon)\, ds$$

$$\mathcal{T}(\upsilon)y(t) = g(t,\upsilon) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, y(s,\upsilon), \upsilon) ds$$

For all $t \in R_+$ & $\upsilon \in \Omega$.

Then the random integral equation transformed into random operator equation

$$y(t,\upsilon) = \mathcal{S}(\upsilon)y(t)\mathcal{T}(\upsilon)y(t) \tag{2}$$

For all $t \in R_+$ & $\upsilon \in \Omega$.

We shall show that the operators $\mathcal{S}(\upsilon),\ \mathcal{T}(\upsilon)$ satisfies all the conditions of the corollary (2.11). This will be done in following steps.

Step – 1: First we show that $\mathcal{S}(\upsilon), \mathcal{T}(\upsilon)$ are random operators on C. By hypothesis H0 the function

$\upsilon \to p(t, y(t,\upsilon), \upsilon)$ is measurable for all $t \in R_+$ & $y \in R$. The product

$(t-s)^{\xi-1}p(s, y(s,\upsilon), \upsilon)$ of continuous and measurable functions is again measurable. We know that Riemann integral as a limit of a finite sum of measurable functions is again measurable. Therefore, the function $\upsilon \to \frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}p(s, y(s,\upsilon), \upsilon)\ ds$ is measurable. Then the function $\upsilon \to \mathcal{S}(\upsilon)y$ is measurable for all $y \in R$. Hence $\mathcal{S}(\upsilon)$ is random operator. By hypothesis H_4, H_5 $\upsilon \to g(t,\upsilon)$ is measurable and $\upsilon \to q(t, y, \upsilon)$ is measurable for all $t \in R_+$, $y \in R$. The product

$(t-s)^{\zeta-1}q(s, y(s,\upsilon), \upsilon)$ of continuous and measurable functions is again measurable. We know that Riemann integral as a limit of a finite sum of measurable function is again measurable. Therefore, the function $\upsilon \to \frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}q(s, y(s,\upsilon), \upsilon)\ ds$ is measurable. Since $\mathfrak{B}$ is separable so C is also separable, the sum of two measurable functions is measurable. Consequently, the function

$\upsilon \to g(t,\upsilon) + \frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}q(s, y(s,\upsilon), \upsilon)\ ds$ is measurable. Then the function $\upsilon \to \mathcal{T}(\upsilon)y$ is measurable for all $y \in R$. Hence $\mathcal{T}(\upsilon)$ is random operator on C.

Step – II: Next we show that $\mathcal{S}(\upsilon)$ is Lipschitz random operator on C.

Let $y, z \in C$

$$|\mathcal{S}(\upsilon)y(t) - \mathcal{S}(\upsilon)z(t)|$$

$$= \left|\frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}p(s, y(s,\upsilon), \upsilon)\ ds - \frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}p(s, z(s,\upsilon), \upsilon)\ ds\right|$$

$$\le \frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\ |p(s, y(s,\upsilon), \upsilon) - p(s, z(s,\upsilon), \upsilon)|\ ds$$

$$\le \frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\ \mathcal{L}(s,\upsilon)|y(s,\upsilon) - z(s,\upsilon)|\ ds$$

Taking the supremum over t in the above inequality, we obtain

$$\|\mathcal{S}(\upsilon)y - \mathcal{S}(\upsilon)z\| \le \|y(\upsilon) - z(\upsilon)\|\ sup\left(\frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\ \mathcal{L}(s,\upsilon)\ ds\right)$$

$$\le K_2\|y(\upsilon) - z(\upsilon)\|$$

$$\because K_2 = sup\left(\frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\ \mathcal{L}(s,\upsilon)\ ds\right)$$

$$\|\mathcal{S}(\upsilon)y - \mathcal{S}(\upsilon)z\| \le K_2\ \|y(\upsilon) - z(\upsilon)\|$$

$\mathcal{S}(\upsilon)$ is Lipschitz with Lipschitz constant K_2 .

Step – III: Now, show that $\mathcal{T}(\upsilon)$ is continuous on C.

Let $\{y_n\}$ be convergent sequence of points in C converging to the point $y \in C$, then it is enough to prove that

$$\lim_{n\to\infty} \mathcal{T}(v)y_n(t) = \mathcal{T}(v)y(t), \quad t \in R_+$$

By Lebesgue dominated converging theorem we obtain

$$\lim_{n\to\infty} \mathcal{T}(v)y_n(t) = \lim_{n\to\infty}\{g(t,v) + \frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1} q(s, y_n(s,v), v)\ ds\}$$

$$= \left\{g(t,v) + \frac{1}{\Gamma\zeta}\int_0^t \lim_{n\to\infty}(t-s)^{\zeta-1} q(s, y_n(s,v), v) ds\right\}$$

$$= \left\{g(t,v) + \frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}\ q(s, y(s,v), v)\ ds\right\}$$

$$= \mathcal{T}(v)y(t)$$

For every $t \in R_+$ and $v \in \Omega$.

This shows that the function $\mathcal{T}(v)$ is continuous on C.

Step – IV: To show that $\mathcal{T}(v)$ is compact operator on C.

It suffices to show that$\mathcal{T}(v)(C)$ is uniformly bounded and equicontinuous set in $\mathfrak{B}$, for each $v \in \Omega$. First, we show that $\mathcal{T}(v)(C)$ is uniformly bounded for each $v \in \Omega$.

Let $y \in C$ be arbitrary thus by hypothesis H_6 q is $L^1(v)$-caratheodory

$$|\mathcal{T}(v)y(t)| = \left|g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ q(s, y(s,v), v) ds\right|$$

$$\leq |g(t,v)| + \left|\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ q(s, y(s,v), v) ds\right|$$

$$\leq |g(t,v)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ |q(s, y(s,v), v)|\ ds$$

$$\leq |g(t,v)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ \psi_r(s,v)\ ds$$

$$\leq |g(t,v)| + \frac{1}{\Gamma(\zeta)}\ \mathcal{V}(t,v)$$

Taking supremum all over t

$\|\mathcal{T}(v)y\| \leq \|g(v)\| + K_1$ for all $t \in R_+$

Hence $\mathcal{T}(v)(C)$ is uniformly bounded subset of $\mathfrak{B}$.

Step – V: Now we shall show that $\mathcal{T}(v)(C)$ is equicontinuous set in $\mathfrak{B}$.

Let $t_1, t_2 \in R_+$ with $t_1 > t_2$

$|\mathcal{T}(v)y(t_1) - \mathcal{T}(v)y(t_2)|$

$$\leq \left|g(t_1,v) + \frac{1}{\Gamma(\zeta)}\int_0^{t_1} (t_1-s)^{\zeta-1}\ q(s, y(s,v), v) ds - (g(t_2,v) - \frac{1}{\Gamma(\zeta)}\int_0^{t_2} (t_2-s)^{\zeta-1}\ q(s, y(s,v), v) ds)\right|$$

$\leq |g(t_1,\upsilon)-(g(t_2,\upsilon)|$

$+\left|\frac{1}{\Gamma(\zeta)}\int_0^{t_1}(t_1-s)^{\zeta-1}\, q(s,y(s,\upsilon),\upsilon)ds-\frac{1}{\Gamma(\zeta)}\int_0^{t_2}(t_2-s)^{\zeta-1}\, q(s,y(s,\upsilon),\upsilon)ds\right|$

$\leq |g(t_1,\upsilon)-(g(t_2,\upsilon)|$

$+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_1}(t_1-s)^{\zeta-1}\, q(s,y(s,\upsilon),\upsilon)ds-\int_0^{t_2}(t_2-s)^{\zeta-1}\, q(s,y(s,\upsilon),\upsilon)ds\right|$

$\leq |g(t_1,\upsilon)-(g(t_2,\upsilon)|+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_1}(t_1-s)^{\zeta-1}\,\psi_r(s,\upsilon)ds-\int_0^{t_2}(t_2-s)^{\zeta-1}\,\psi_r(s,\upsilon)ds\right|$

$\leq |g(t_1,\upsilon)-(g(t_2,\upsilon)|+\frac{1}{\Gamma\zeta}|\mathcal{V}(t_1,\upsilon)-\mathcal{V}(t_2,\upsilon)|$

Since $\mathcal{V}(t,\upsilon)=\int_0^t(t-s)^{\zeta-1}\,\psi_r(s,\upsilon)\,ds$ and $\mathcal{V}$ is uniformly continuous function

Hence, $|\mathcal{T}(\upsilon)y(t_1)-\mathcal{T}(\upsilon)y(t_2)|\to 0$ as $t_1\to t_2$ for all $t_1,t_2\in R_+$ and $\upsilon\in\Omega$

This shows that $\mathcal{T}(\upsilon)(C)$ is equicontinuous set in $\mathfrak{B}$.

Hence $\mathcal{T}(\upsilon)(C)$ is equicontinuous set in $\mathfrak{B}=BM(R_+,R)$ and so by the Arzela-Ascoli theorem $\mathcal{T}(\upsilon)(C)$ is relatively compact. $\mathcal{T}(\upsilon)$ is continuous and compact operator on C.

$\therefore\mathcal{T}(\upsilon)$is completely continuous operator on C.

Step VI: To show that the third condition of the corollary (2.11) is satisfied

Let $y\in C$ be arbitrary elements such that

$y=\mathcal{S}(\upsilon)y\,\mathcal{T}(\upsilon)y$

Then we have

$|\mathcal{S}(\upsilon)y(t)\mathcal{T}(\upsilon)y(t)|$

$=\left|\left[\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}\,p(s,y(s,\upsilon),\upsilon)\,ds\right]\left[g(t,\upsilon)+\frac{1}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\,q(s,y(s,\upsilon),\upsilon)ds\right]\right|$

$$=\left|\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}p(s,y(s,\upsilon),\upsilon)\,ds\right|\left|g(t,\upsilon)+\frac{1}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\,q(s,y(s,\upsilon),\upsilon)ds\right|$$

$\leq\left|\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}\,p(s,y(s,\upsilon),\upsilon)\,ds\right|\left[|g(t,\upsilon)|+\left|\frac{1}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\,q(s,y(s,\upsilon),\upsilon)ds\right|\right]$

$\leq\left[\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}\,|p(s,y(s,\upsilon),\upsilon)|\,ds\right]$

$\times\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\,|q(s,y(s,\upsilon),\upsilon)|\,ds\right]$

$\leq\left[\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}\phi(s,\upsilon)\,ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\psi_r(s,\upsilon)\,ds\right]$

$\leq\left[\frac{1}{\Gamma(\xi)}\int_0^t(t-s)^{\xi-1}\phi(s,\upsilon)\,ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\,\mathcal{V}(t,\upsilon)\right]$

Taking supremum over t

$$\|\mathcal{S}(v)y\,\mathcal{T}(v)y\| \le \phi_1[\|g(v)\| + K_1] = r$$

$$\because \phi_1 = \sup\{\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,v)\ ds\}$$

$$\therefore \mathcal{S}(v)y\,\mathcal{T}(v)y \in C$$

Step VII: Finally, we show that

$$\begin{aligned}
\mathcal{M}(v) &= \|\mathcal{T}(v)(c)\| \\
&= \sup_{a\in C}\{\|\mathcal{T}(v)a\| : a \in C\} \\
&= \sup_{a\in C}\left\{\sup_{t\in R_+}|\mathcal{T}(v)a(t)|\right\} \\
&= \sup_{a\in C}\left\{\sup_{t\in R_+}\left|g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,y(s,v),v)ds\right|\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\left[|g(t,v)| + \left|\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,y(s,v),v)ds\right|\right]\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\left[|g(t,v)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, |q(s,y(s,v),v)|\, ds\right]\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\left[|g(t,v)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\psi_r(s,v)\, ds\right]\right\} \\
&\le \sup\left\{\sup_{t\in R_+}\left[|g(t,v)| + \frac{1}{\Gamma(\zeta)}\,\mathcal{V}(t,v)\right]\right\} \\
&\le \|g(v)\| + K_1
\end{aligned}$$

And therefore $\beta(v)\,\mathcal{M}(v) < K_2\,(\|g(v)\| + K_1) < 1$

Thus, all the conditions of the corollary (2.11) are satisfied and hence the operator equation $y = \mathcal{S}(v)y\,\mathcal{T}(v)y$ has random solutions in C. Hence FONRIE (1) has random solutions in C.

Step VIII: Finally, we have to show that the locally attractivity of the solutions for FONRIE.

Let y & z be two solutions FONRIE in C. Then we have

$$|y(t,v) - z(t,v)| = \left|\begin{matrix}\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}p(s,y(s,v),v)\ ds\right]\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,y(s,v),v)ds\right] \\ -\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\, p(s,z(s,v),v)\ ds\right]\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,z(s,v),v)ds\right]\end{matrix}\right|$$

$$\le \left|\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\, p(s,y(s,v),v)\ ds\right]\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,y(s,v),v)ds\right]\right| + \left|\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\, p(s,z(s,v),v)\ ds\right]\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,z(s,v),v)ds\right]\right|$$

$$\le \left|\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\, p(s,y(s,v),v)\ ds\right|\left|g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s,y(s,v),v)ds\right|$$

$$+\left|\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1} p(s,z(s,\upsilon),\upsilon)\ ds\right|\left|g(t,\upsilon)+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ q(s,z(s,\upsilon),\upsilon)ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\ p(s,y(s,\upsilon),\upsilon)\ ds\right|\left[|g(t,\upsilon)|+\left|\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ q(s,y(s,\upsilon),\upsilon)ds\right|\right]+$$
$$\left|\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1} p(s,z(s,\upsilon),\upsilon)\ ds\right|\left[|g(t,\upsilon)|+\left|\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ q(s,z(s,\upsilon),\upsilon)ds\right|\right]$$

$$\leq \left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\ |p(s,y(s,\upsilon),\upsilon)|\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ |q(s,y(s,\upsilon),\upsilon)|\ ds\right]$$

$$+\ \left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\ |p(s,z(s,\upsilon),\upsilon)|\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\ |q(s,z(s,\upsilon),\upsilon)|\ ds\right]$$

$$\leq\ \left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,\upsilon)\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\psi_r(s,\upsilon)\ ds\right]$$

$$+\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,\upsilon)\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1} q_r(s,\upsilon)\ ds\right]$$

$$\leq 2\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,\upsilon)\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\psi_r(s,\upsilon)\ ds\right]$$

$$\leq 2\left[\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,\upsilon)\ ds\right]\left[|g(t,\upsilon)|+\frac{1}{\Gamma(\zeta)}\mathcal{V}(t,\upsilon)\right]$$

Taking limit in above inequality as $t\to\infty$ yields

$$\lim_{t\to\infty}|y(t,\upsilon)-z(t,\upsilon)| = 2[\lim_{t\to\infty}\frac{1}{\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}\phi(s,\upsilon)\ ds]\ [\lim_{t\to\infty}\ |g(t,\upsilon)|+\lim_{t\to\infty}\frac{1}{\Gamma(\zeta)}\mathcal{V}(t,\upsilon)]$$

$$\lim_{t\to\infty}|y(t,\upsilon)-z(t,\upsilon)| = 0$$

$\because \lim_{t\to\infty}\ |g(t,\upsilon)| = 0$ And $\lim_{t\to\infty}\mathcal{V}(t,\upsilon) = 0$

Hence Random solutions of operator equation (2) are locally attractive.

4. Existence of extremal random solutions:

In this part, we established the existence of extremal solutions under a few reasonable monotonicity requirements.

4.1 Definition: (Dhage 2007; Karande 2014) A non-empty closed subset K of a separable Banach algebra $\mathfrak{B}$ is called cone if

(i) $K+K\subseteq K$
(ii) $\delta K\subset K\ for\ all\ \ \delta\in R_+$
(iii) $\{-K\}\cap K=\{0\}$

Where 0 is zero element of $\mathfrak{B}$

We define order relation $\leq$ in $\mathfrak{B}$ with the help of a cone K as follows

Let $x,y\in\mathfrak{B}$ then $x\leq y$ iff $y-x\in K$

The Banach algebra $\mathfrak{B}$ with order relation $\leq$ is called ordered Banach algebra and it is denoted by $(\mathfrak{B}, \|.\|, \leq)$.

A cone K in $\mathfrak{B}$ is known as normal if the norm $\|.\|$ is defined on $\mathfrak{B}$ is monotone increasing on K, i.e. if $x, y \in K$ with $x \leq y$ then there exist a constant $N > 0$ such that

$$\|x\| \leq N\|y\|$$

It is known that if the cone K is normal in $\mathfrak{B}$ then every order-bounded set in $\mathfrak{B}$ is nor- bounded set in $\mathfrak{B}$.

It is known that if the cone K in $\mathfrak{B}$ is normal, then every order interval is norm bounded set in $\mathfrak{B}$. Let $a, b: \Omega \to \mathfrak{B}$ be two measurable functions then by $a \leq b$ we mean $a(v) \leq b(v)$ for all $v \in \Omega$. In this case the random order interval $[a, b]$ is defined to be a set in $\mathfrak{B}$ given by

$$[a, b] = \{x \in \mathfrak{B} : a(v) \leq x \leq b(v), for\ all\ v \in \Omega\}$$

$$= \bigcap_{v \in \Omega} [a(v), b(v)]$$

4.2 Definition: (Dhage 2007; Karande 2014) A random operator $T: \Omega \times \mathfrak{B} \to \mathfrak{B}$ is called monotone non-decreasing if for any $x, y \in K$, $x \leq y$ implies $T(v)x \leq T(v)y$ for all $v \in \Omega$.

4.3 Definition: (Dhage 2007) Define order relation $\leq$ in continuous real valued space $C(R_+, R)$ by a cone K in $C(R_+, R)$ by

$$K = \{x \in c(R_+, R): x \geq 0\}$$

Clearly cone K is normal in $c(R_+, R)$.

It is known that if the cone K in $\mathfrak{B}$ is normal, then every order interval is norm bounded set in $\mathfrak{B}$

4.4 Theorem: (Dhage 2007) Let $\mathcal{S}, \mathcal{T} : \Omega \times \mathfrak{B} \to \mathfrak{B}$ be two monotonic non decreasing random operators satisfying for each $v \in \Omega$

(a) $\mathcal{S}(v)$ is Lipschitz with the Lipschitz constant $\alpha(v)$
(b) $\mathcal{T}(v)$ is completely continuous
(c) There exist two continuous functions $a, b : \Omega \to \mathfrak{B}$ s. t..

$$a(v) \leq \mathcal{S}(v)a\,\mathcal{T}(v)a \quad \& \quad \mathcal{S}(v)b\ \mathcal{T}(v)b \leq b(v)$$

Further cone K in $\mathfrak{B}$ is normal then the random equation

$$\mathcal{S}(v)y\mathcal{T}(v)y = y$$

Has the least random solutions y_m and the greatest random solution y_M in $[a, b]$ whenever

$$\alpha(v)M(v) < 1 \text{ for each } v \in \Omega,$$

Where, $M(v) = \|\mathcal{T}(v)([a, b])\|$

Moreover, $y_m(v) = \lim_{n\to\infty} x_n(v)$; $y_M(v) = \lim_{n\to\infty} y_n(v)$

Where, $x_{n+1}(v) = \mathcal{S}(v)x_n\mathcal{T}(v)x_n,\ n \geq 0$

With $x_0(v) = a(v)$

$$y_{n+1}(v) = \mathcal{S}(v)y_n\mathcal{T}(v)y_n, \ n \geq 0$$

With $y_0(v) = b(v)$

We consider the following hypothesis

(B_0) The function $p(t,y,v), g(t,v)$ and $q(t,y,v)$ are nondecreasing a.e. for $t \in R_+$

(B_1) The function $v \to p(t,y,v)$ is measurable, the function $g: \Omega \to C(R_+, R)$ be measurable and the function $v \to q(t,y,v)$ is measurable for all $t \in R_+, y \in R$.

(B_2) The function $p: R_+ \times R \times \Omega \to R - \{0\}$ is continuous and there exist a function $\mathcal{L}: \Omega \to L^1(R_+, R)$ such that for each $v \in \Omega$

$$|p(t,x(t,v),v) - p(t,y(t,v),v)| \leq \mathcal{L}(t,v)|x(t,v) - y(t,v)| \ a.e.\ t \in R_+, v \in \Omega,$$

$x, y \in R$.

(B_3) The function q is $L^1(v)$-caratheodory

(B_4) The FONRIE has a lower random solution a and upper random solution b with $a \leq b$.

Remark: Note that if hypothesis B_1, B_3, H_4 holds then there exist constant $K_1 > 0$ such that

$$K_1 = \sup\{\frac{1}{\Gamma\zeta}\mathcal{V}(t,v) : t \in R_+ \ \&\ v \in \Omega\}$$

4.5 Theorem: Assume that the hypothesis B_0-B_4 hold then FONRIE (1) has minimal random solution y_m and maximal random solution y_M in $[a,b]$ whenever

$$K_2(\|g(v)\| + K_1) < 1$$

Moreover $y_m(v) = \lim_{n\to\infty} x_n(v)$; $y_M(v) = \lim_{n\to\infty} y_n(v)$

Where, $x_{n+1}(t,v) = \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, p(s, x_n(s,v), v)\ ds\right]$

$$\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, x_n(s,v), v)ds\right]$$

. for all $t \in R_+$ with $x_0 = a$

$$y_{n+1}(t,v) = \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, p(s, y_n(s,v), v)\ ds\right]$$

$$\times \left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, y_n(s,v), v)ds\right]$$

. for all $t \in R_+$ with $y_0 = b$

Proof: Let $\mathfrak{B} = BM(R_+, R)$ be separable Banach space and we define an order relation $\leq$ by the cone K.

Clearly cone K is normal in $\mathfrak{B}$ .Define the operators $\mathcal{S}, \mathcal{T}$ by

$$\mathcal{S}(v)y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} p(s, y(s,v), v)\, ds$$

$$\mathcal{T}(v)y(t) = g(t,v) + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1}\, q(s, y(s,v), v) ds$$

Then the random integral equation equivalent to random operator

$$y(t,v) = \mathcal{S}(v)y(t)\mathcal{T}(v)y(t) \quad \text{for all } t \in R_+$$

We have shown that the operators $\mathcal{S}(v)$ and $\mathcal{T}(v)$ satisfies theorem (3.2). In the theorem we have proved that $\mathcal{S}(v)$ is Lipschitz with Lipschitz constant K_2 and $\mathcal{T}(v)$ is completely continuous random operator.

In theorem we have proven that $\mathcal{S}(v)$ is Lipschitz and $M(v) = \|\mathcal{T}(v)([a,b])\|$.

To prove that $\mathcal{S}(v)$ & $\mathcal{T}(v)$ are monotone nondecreasing on $[a,b]$

Let $x, y \in [a,b]$ be such that $x \leq y$ then

$$\begin{aligned}\mathcal{S}(v)x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} p(s, x(s,v), v)\, ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} p(s, y(s,v), v)\, ds \\ &\leq \mathcal{S}(v)y(t)\end{aligned}$$

For all $t \in R_+$

Similarly,

$$\begin{aligned}\mathcal{T}(v)x(t) &= g(t,v) + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1}\, q(s, x(s,v), v) ds \\ &\leq\ g(t,v) + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1}\, q(s, y(s,v), v) ds \\ &\leq \mathcal{T}(v)y(t)\end{aligned}$$

For all $t \in R_+$

Thus, random operators $\mathcal{S}(v)$ and $\mathcal{T}(v)$ are monotone non-decreasing operators on $[a,b]$ and hypothesis B_4, a is lower random solution to FONRIE we have

$$\begin{aligned}a(t,v) &\leq \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, p(s, a(s,v), v)\ ds\right] \\ &\quad \times \left[g(t,v) + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1}\, q(s, a(s,v), v) ds\right] \\ &\leq \mathcal{S}(v)a(t)\mathcal{T}(v)a(t)\end{aligned}$$

For all $t \in R_+$ and $v \in \Omega$

b is upper random solution of FONRIE we have

$$\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, p(s, b(s,v), v)\ ds\right]$$

$$\times\left[g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, b(s,v), v) ds\right] \le b(t,v)$$

$$\mathcal{S}(v)b(t)\mathcal{T}(v)b(t) \le b(t,v)$$

For all $t \in R_+$ and $v \in \Omega$

$$\therefore\ \mathcal{S}(v)y(t)\mathcal{T}(v)y(t) \in [a,b]$$

For all $y \in [a,b]$

Now,

$$M(v) = \|\mathcal{T}(v)([a,b])\|$$

$$= sup\{\|\mathcal{T}(v)y\|: y \in [a,b]\}$$

$$= \underset{y\in[a,b]}{sup}\left\{\underset{t\in R_+}{sup}|\mathcal{T}(v)y(t)|\right\}$$

$$= \sup\left\{\underset{t\in R_+}{sup}\left|g(t,v) + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\, q(s, y(s,v), v) ds\right|\right\}$$

$$= \sup\left\{\underset{t\in R_+}{sup}\left(|g(t,v)| + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}|q(s, y(s,v), v)|\ ds\right)\right\}$$

$$\le \|g(v)\| + K_1$$

And therefore $\alpha(v)M(v) = K_2(\|g(v)\| + K_1) < 1$

For each $v \in \Omega$

Hence by theorem (4.4) FONRIE has a minimal random solution y_m and a maximal random solution y_M in $[a,b]$

Moreover,

$$y_m(v) = \lim_{n\to\infty} x_n(v)\ \ ;$$

$$y_M(v) = \lim_{n\to\infty} y_n(v)$$

Where $\{x_{n+1}(v)\}$ and $\{y_{n+1}(v)\}$ are defined by theorem.

Example: Let $v \in \Omega = [0,1]$ and $\xi = \frac{2}{3}, \zeta = \frac{1}{3}$ with FONRIE

$$y(t,v) = \left[\frac{1}{\Gamma\left(\frac{2}{3}\right)}\int_0^t (t-s)^{\left(\frac{2}{3}-1\right)} e^{-5s}(1+2v)\tanh y\ ds\right]$$

$$\times\left[v\frac{e^{-3t}}{20} + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_0^t (t-s)^{\frac{1}{3}-1}\, e^{\frac{-s}{2}}\,(1+v^2)\,\frac{tan^{-1}(y)}{\pi/2} ds\right] \quad (3)$$

Solution: Here,

$p(t,y,v) = e^{-5t}(1+2v)\tanh y$,

$q(t,y,v) = e^{\frac{-t}{2}}(1+v^2)\frac{tan^{-1}(y)}{\pi/2}$ and $g(t,v) = v\frac{e^{-3t}}{20}$

H_0: For fixed t, y the map $v \to e^{-5t}(1+2v)\tanh y$ is polynomial in v, hence measurable.

H_1: for fixed y, v the factor $e^{-5t}(1+2v)\tanh y$ is continuous in t, hence Rieman integrable.

H_2: $|p(t,y(t,v),v) - p(t,z(t,v),v)| = e^{-5t}(1+2v)|\tanh y - \tanh z|$

$$\leq e^{-5t}(1+2v)|y-z|$$

$$\therefore \mathcal{L}(t,v) = e^{-5t}(1+2v) \in L^1(R_+,R)$$

H_3: $p(t,y,v) = e^{-5t}(1+2v)\tanh y$

$|p(t,y,v)| \leq e^{-5t}(1+2v)$

So $\phi(t,v) = e^{-5t}(1+2v) \in L^1(R_+,R)$

H_4: $g(t,v) = v\frac{e^{-3t}}{20}$ is continuous in t measurable in v.

$|g(t,v)| \leq \frac{e^{-3t}}{20}$ vanishes as $t \to \infty$

Now $\|g(v)\| = \frac{1}{20} = 0.05$

H_5: $q(t,y,v) = e^{\frac{-t}{2}}(1+v^2)\frac{tan^{-1}(y)}{\pi/2}$ for fixed t, y (t,y,v) is polynomial in v hence $q(t,y,v)$ is measurable.

H_6: $\left|\frac{tan^{-1}(y)}{\pi/2}\right| = \frac{|tan^{-1}(y)|}{\pi/2} \leq 1$

So $|q(t,y,v)| \leq e^{\frac{-t}{2}}(1+v^2)$ hence $\psi_r(t,v) = e^{\frac{-t}{2}}(1+v^2) \in L^1(R_+,R)$

Now, $K_1 = \sup\{\frac{1}{\Gamma(\zeta)}\mathcal{V}(t,v): t \in R_+, v \in \Omega\}$

where $\mathcal{V}(t,v) = \int_0^t (t-s)^{\frac{1}{3}-1} e^{\frac{-s}{2}}(1+v^2)ds \leq 2\int_0^t (t-s)^{\frac{1}{3}-1} e^{\frac{-s}{2}} ds$

$\therefore K_1 = \sup\{\frac{2}{\Gamma(\zeta\frac{1}{3})}\int_0^t (t-s)^{\frac{1}{3}-1} e^{\frac{-s}{2}} ds\}$

Put $I_1(t) = \int_0^t (t-s)^{\frac{1}{3}-1} e^{\frac{-s}{2}} ds$

Put $t - s = n \Rightarrow dn = -ds$

$$I_1(t) = \int_0^t n^{\frac{1}{3}-1} e^{\frac{-(t-n)}{2}} ds = e^{-\frac{t}{2}}\int_0^t n^{\frac{-2}{3}} e^{\frac{n}{2}} dn$$

$$\sup I_1(t) \approx 1.4860$$

$\therefore\ K_1 \approx \frac{2}{2.6789} \times 1.4863 \approx 1.1096$

And $K_2 = \sup\{\frac{1}{\Gamma\xi}\int_0^t (t-s)^{\xi-1}\,\mathcal{L}(s,\upsilon)\,ds\,\}$

$$= \sup\{\frac{1}{\Gamma\frac{2}{3}}\int_0^t (t-s)^{\frac{2}{3}-1}\,e^{-5s}(1+2\upsilon)\,ds\,\}$$

$$= \sup\{\frac{3}{\Gamma\frac{2}{3}}\int_0^t (t-s)^{\frac{-1}{3}}\,e^{-5s}\,ds\,\}$$

Let $I_2 = \int_0^t (t-s)^{\frac{-1}{3}}\,e^{-5s}\,ds$

$$= \int_0^t m^{\frac{-1}{3}}\,e^{-5(t-m)}\,dm \; = e^{-5t}\int_0^t m^{\frac{-1}{3}}\,e^{5m}\,dm$$

$$\sup I_2(t) \approx 0.3036$$

$$K_2 \approx \frac{3}{1.3541} \times 0.3036 \approx 0.6726$$

$\therefore K_2(\|g(\upsilon)\| + K_1) \approx 0.6726(0.05 + 1.1096) \approx 0.7799 < 1$ hence from H_0-H_6 and theorem (3.2) the equation (3) has random solutions and these are locally attractive.

5. **Conclusion**

In this study, we demonstrated the existence and local attractivity of random solutions using a hybrid fixed-point theorem under Lipschitz and Caratheodory conditions. Also, we have discussed extremal solutions. The illustrative example confirmed the applicability of the theoretical results and demonstrate the effectiveness of the proposed approach. This example is combining the theory and application which gives ideas for future research in related areas. The findings of this study extend a number of existing theories for deterministic and non-fractional random integral equations to a more general stochastic fractional context. Further research may focus on extending these results to more general classes of random fractional integro-differential systems and exploring numerical method for their approximation.

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