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Further Congruences For $[j, k]$ − Overpartition with Even Parts Distinct

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Abstract:

Naika, MS Mahadeva, Harishkumar T, and T. N. Veeranayaka [1] defined $\overline{ped}_{j,k}(n)$ as the number of [j, k]- overpartitions of a integer $n \geq 0$ with distinct even parts and with the restriction that the first occurrence of each distinct part congruent to $j\ modulo\ k$ may be overlined. This paper presents some infinite families of congruence modulo powers of 2 for $\overline{ped}_{3,3}(n)$ and $\overline{ped}_{3,6}(n)$. For example,

$$\overline{ped}_{3,3}\left(6.p^{2\alpha+1}.(pn+w)+\frac{13.p^{2\alpha+2}-1}{4}\right)=0\ (mod\ 8)$$
$$\overline{ped}_{3,6}(24.p^{2\alpha+1}.(pn`+m)+13.p^{2\alpha+2})q^n\equiv 0\quad(mod\ 4)$$

Keywords: Congruences, Partition, Generating function, Overpartition, [j, k]-overpartition

1. Introduction

A partition of a positive integer n is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots\ldots \geq \lambda_m$ of positive integers such that

$$\lambda_1+\lambda_2+\lambda_3+\ldots\ldots\ldots+\lambda_m=n$$

It is denoted by $p(n)$ with $p(0) = 1$ and the generating function is given by

$$\sum_{n=0}^{\infty} p(n)q^n=\frac{1}{(q;q)_\infty}=\frac{1}{f_1}$$

where

$$f_k=(q^k;q^k)_\infty=\prod_{n=1}^{\infty}(1-q^{kn}),\qquad |q|<1,\quad k\geq 1.$$

Ramanujan [2] [3] studied the arithmetic properties of $p(n)$ and established the following three congruences

$$p(5n+4)\equiv 0(mod 5)$$

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$$p(7n+5) \equiv 0 (mod 7)$$

$$p(11n+6) \equiv 0 (mod 11)$$

An overpartition [4] of an integer $n \geq 0$ represented by $\bar{p}(n)$ is a partition with the first occurrence of a part may be overlined. For example, $\bar{p}(2) = 4$ given by

$$2, \quad \bar{2}, \quad 1+1, \quad \bar{1}+1$$

Corteel, S., & Lovejoy, J. [4] expressed the generating function as follows

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{f_2}{f_1^2}$$

In [1], Naika, M. S., et al. defined $[j,k]$ −overpartitions of an integer $n \geq 0$ denoted by $\overline{ped}_{j,k}(n)$ with distinct even parts and the first occurrence of each distinct part congruent to j $modulo$ k may be overlined. The authors defined the generating function as

$$\sum_{n=0}^{\infty} \overline{ped}_{j,k}(n) q^n = \frac{(q^4;q^4)_\infty (-q^j;q^k)_\infty}{(q;q)_\infty} \tag{1.1}$$

2. Preliminaries

In this section, some important notations, identities and important lemmas, which will be used to establish the results will be discussed

Ramanujan's general theta function [[5], p.34, Equation 18.1] is

$$f(a,b) = \sum_{n=0}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1$$

Using Jacobi's Triple product identity [[5], p.35, Entry 19], we have

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty$$

Three special cases of $f(a,b)$ are [[5], p.36, Entry22]

$$\phi(q) = f(q,q) = \sum_{n=0}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{f_2^2}{f_1^2}$$

$$\boldsymbol{f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}} = f_1}$$

Lemma 2.1. [6] *The following 2-dissections holds*:

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \tag{2.1}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8} \tag{2.2}$$

Proof of Lemma 2.1, can be found in Berndt [[6], Entry 25, p. 40].

Lemma 2.2 [7] *The following 2-dissections holds*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4} \tag{2.3}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \tag{2.4}$$

Proof. (2.3) and (2.4) are equivalent to Eq. (22.7.5) and (30.9.9) of Hirschhorn [7], respectively.

Lemma 2.3 *We have the 3-dissection*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3} \tag{2.5}$$

Proof. Lemma. 2.3 follows from [8] Theorem 3.1 and [9] Lemma.2.6

Lemma 2.4 [10] For *any odd prime p, we have*

$$\psi(q) = \sum_{k=0}^{\infty} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi\left(q^{p^2}\right)$$

Moreover, for, $0 \le k \le \frac{p-1}{2}$

$$\frac{k^2+k}{2} \neq \frac{p^2-1}{8}\ (mod\ 8)$$

Proof. For proof of the Lemma see [10], Theorem 2.1

Lemma 2.5 [10] *For each prime, p* ≥ 5, *we have*

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}$$

Where $\frac{\pm p-1}{6} = \begin{cases} \frac{p-1}{6} & p \equiv 1\ (mod\ 6) \\ \frac{-(p-1)}{6} & p \equiv -1\ (mod\ 6) \end{cases}$

Furthermore, if $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2}$ *and* $k \neq \frac{\pm p-1}{6}$,

Then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

Proof. For the proof of the Lemma See [10], Theorem 2.2.

Lemma 2.6. *Givena prime* p *and integers* $j \geq 0, k \geq 0$ *and* $m \geq 0$, *we have*

$$f_k^{p^j m} \equiv f_{pk}^{p^{j-1} m} \pmod{p^j}$$

For any odd prime p and a positive integer a relative prime to p, the *Legendre symboll* $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & if\ a\ is\ a\ quadratic\ residue\ of\ p \\ -1, & if\ a\ is\ a\ quadratic\ non-residue\ of\ p \end{cases}$$

3. CONGRUENCES FOR $\overline{ped}_{3,3}(n)$

Theorem. 3.1 *Suppose* p *is a prime and* $\left(\frac{-12}{p}\right) = -1$. *For* $1 \leq w \leq p-1$, *and any integer* $\alpha \geq 0$,*we have*

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\alpha}.n + \frac{13.p^{2\alpha}-1}{4}\right) q^n = 4 f_1\, \psi(q^4) \pmod{8} \quad (3.1)$$

$$\overline{ped}_{3,3}\left(6.p^{2\alpha+1}.(pn+w) + \frac{13.p^{2\alpha+2}-1}{4}\right) = 0 \pmod{8} \quad (3.2)$$

Proof. Setting $j = 3$ and $k = 3$ in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(n) q^n = \frac{f_4 f_6}{f_1 f_3} \quad (3.3)$$

Making use of (2.5) in (3.3), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(n) q^n = \frac{f_6}{f_3}\left(\frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}\right) \quad (3.4)$$

From (3.4), collecting the terms of the form q^{3n}, q^{3n+1} and q^{3n+2} and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(3n) q^n = \frac{f_2 f_4 f_6^4}{f_1^4 f_{12}^2} \quad (3.5)$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(3n+1) q^n = \frac{f_2^3 f_3^3 f_{12}}{f_1^5 f_6^2} \quad (3.6)$$

$$and \quad \sum_{n=0}^{\infty} \overline{ped}_{3,3}(3n+2)q^n = 2\,\frac{f_3^2 f_6 f_{12}}{f_1^4} \tag{3.7}$$

Using (2.1) in (3.5), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(3n)q^n = \frac{f_4^{15} f_6^4}{f_2^{13} f_8^4 f_{12}^2} + 4q\,\frac{f_4^3 f_6^4 f_8^4}{f_2^9 f_{12}^2}$$

Extracting the terms that include q^{2n+1} replacing q with $q^{\frac{1}{2}}$, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(6n+3)q^n \equiv 4\,\frac{f_2^3 f_3^4 f_4^4}{f_1^9 f_6^2}\ (mod\ 8)$$

Using Lemma. 2.6 for $p = 2\ and\ j = 3$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(6n+3)q^n \equiv 4\, f_1\, f_4^3 (mod\ 8)$$

$$\equiv 4\, f_1\, \psi(q^4)\ (mod\ 8) \tag{3.8}$$

(3.8) is the $\alpha = 0$ case of (3.1). Suppose (3.1) holds for some $\alpha \geq 0$. Using Lemma. 2.4 and Lemma. 2.5 in (3.1), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\alpha}.n + \frac{13.p^{2\alpha}-1}{4}\right)q^n$$

$$\equiv 4\left[\sum_{\substack{x=-\frac{p-1}{2} \\ x\neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^x q^{\frac{3x^2+x}{2}} f\left(-q^{\frac{3p^2+(6x+1)p}{2}}, -q^{\frac{3p^2-(6x+1)p}{2}}\right)\right.$$

$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\right] X \left[\sum_{n=0}^{\infty} q^{2(y^2+y)} f(q^{2(p^2+(2y+1)p)}, q^{2(p^2-(2y+1)p)})\right.$$

$$\left. + q^{\frac{p^2-1}{2}} \psi\left(q^{4p^2}\right)\right] (mod\ 8) \tag{3.9}$$

Consider the congruence

$$\frac{3x^2+x}{2} + 2(y^2+y) \equiv \frac{13(p^2-1)}{24}\ (mod\ p)$$

which is equivalent to

$$(6x + 1)^2 + 12\,(2y + 1)^2 \equiv 0 \pmod p \tag{3.10}$$

For $\left(\frac{-12}{p}\right) = -1$, (3.8) have only the solution $x = \frac{(\pm p-1)}{6}$ and $y = \frac{p-1}{2}$. Extracting those terms that involve $q^{pn+\frac{13(p^2-1)}{24}}$ from (3.9), dividing both sides by $q^{\frac{13(p^2-1)}{24}}$, and replacing q with $q^{\frac{1}{p}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\alpha}.\left(pn + \frac{13.(p^2-1)}{24}\right) + \frac{13.p^{2\alpha}-1}{4}\right) q^n \equiv 4\, f_p\, \psi(q^{4p}) \pmod 8 \tag{3.11}$$

which yields

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\alpha+1}.n + \frac{13.p^{2\alpha+2}-1}{4}\right) q^n \equiv 4\, f_p\, \psi(q^{4p}) \pmod 8 \tag{3.12}$$

Again, extracting the terms that include q^{pn}, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\alpha+2}.n + \frac{13.p^{2\alpha+2}-1}{4}\right) q^n \equiv 4\, f_1\, \psi(q^4) \pmod 8$$

Thus (3.1) holds for $\alpha + 1$. Thus (3.1) always holds.

Finally, extracting the terms that include q^{pn+w}, $1 \le w \le p-1$ from (3.12), we obtain (3.2).

Theorem. 3.2 Let p *be any prime such that* $\left(\frac{-6}{p}\right) = -1$, *and* $1 \le u \le p-1$. *If* $\beta \ge 0$ *be an integer then*

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\beta}.n + \frac{5.p^{2\beta}-1}{4}\right) q^n \equiv f_2\, \psi(q) \pmod 4 \tag{3.13}$$

$$\overline{ped}_{3,3}\left(6.p^{2\beta+1}.(pn+u) + \frac{5.p^{2\beta+2}-1}{4}\right) \equiv 0 \pmod 4 \tag{3.14}$$

Proof. From (3.6) and making use of (2.1) and (2.3)

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(3n+1) q^n = \frac{f_2^3 f_{12}}{f_6^2}.\left(\frac{f_3^3}{f_1}\right).\left(\frac{1}{f_1^4}\right)$$

$$= \frac{f_2^3 f_{12}}{f_6^2}.\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right).\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right) \tag{3.15}$$

On extracting only the terms with even power of q from (3.15) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(6n+1)q^n \equiv \frac{f_2^{17}}{f_1^{13} f_4^4} \ (mod\ 4)$$

Thanks to Lemma 2.5 for $p = 2$ and $j = 2$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}(6n+1)q^n \equiv f_2\, \psi\,(q)\ (mod\ 4) \tag{3.16}$$

(3.16) is the $\beta = 0$ case of (3.13). Assume (3.13) holds for some $\beta \geq 0$. Using Lemma. 2.4 and Lemma. 2.5 in (3.13), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\beta}.n + \frac{5.p^{2\beta}-1}{4}\right) q^n$$

$$\equiv \left[\sum_{\substack{x=-\frac{p-1}{2} \\ x \neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^x q^{3x^2+x} f\left(-q^{3p^2+(6x+1)p}, -q^{3p^2-(6x+1)p}\right) \right.$$

$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{2p^2} \right] X \left[\sum_{n=0}^{\infty} q^{\frac{y^2+y}{2}} f(q^{\frac{p^2+(2y+1)p}{2}}, q^{\frac{p^2-(2y+1)p}{2}}) \right.$$

$$\left. + q^{\frac{p^2-1}{8}} \psi\left(q^{p^2}\right) \right] (mod\ 4) \tag{3.17}$$

Consider the congruence

$$(3x^2 + x) + \frac{y^2+y}{2} \equiv \frac{5(p^2-1)}{24} \ (mod\ p)$$

Above congruence yields

$$(12x+2)^2 + 6(\,2y+1)^2 \equiv 0\ (mod\ p)$$

For $\left(\frac{-6}{p}\right) = -1$, above congruence has only solution $x = \frac{\pm p-1}{6}$ and $y = \frac{p-1}{2}$. Extracting terms that include $q^{pn+\frac{5(p^2-1)}{24}}$ from (3.17), dividing by $q^{\frac{5(p^2-1)}{24}}$ and replace q^p with q, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\beta}.(pn + \frac{5(p^2-1)}{24})` + \frac{5.p^{2\beta}-1}{4}\right) q^n \equiv f_{2p}\, \psi\,(q^p)\ (mod\ 4)$$

which implies

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\beta+1}.n` + \frac{5.p^{2\beta+2}-1}{4}\right)q^n \equiv f_{2p}\,\psi\,(q^p)\,(mod\ 4) \tag{3.18}$$

Extracting the terms that include q^{pn} from (3.18) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,3}\left(6.p^{2\beta+2}.n` + \frac{5.p^{2\beta+2}-1}{4}\right)q^n \equiv f_2\,\psi\,(q)\,(mod\ 4)$$

which is the $\beta+1$ case of (3.13). Thus, by induction (3.13) holds for all integer $n \geq 0$ and $\beta \geq 0$.

Also, (3.14) is obtained from (3.18) on extracting q^{pn+v} with $1 \leq v \leq p-1$ from both sides.

4. CONGRUENCE OF $\overline{ped}_{3,6}(n)$

Theorem. 4.1 *Suppose p is any prime such that* $\left(\frac{-4}{p}\right) = -1$ *and* $1 \leq m \leq p-1$. *For any integer* $n \geq 0$ *and* $\delta \geq 0$, *we have*

$$\overline{ped}_{3,6}(12n+7) \equiv 0\ (mod\ 4) \tag{3.19}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.p^{2\delta}.n` + 13.p^{2\delta}\right)q^n \equiv 2\ f_1\,\psi\,(q^4)\ \ (mod\ 4) \tag{3.20}$$

$$and \quad \overline{ped}_{3,6}\left(24.p^{2\delta+1}.(pn` + m) + 13.p^{2\delta+2}\right)q^n \equiv 0\ \ (mod\ 4) \tag{3.22}$$

Proof. Setting $j = 3$ and $k = 6$ in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(n)\,q^n = \frac{f_6^2}{f_3 f_{12}}.\left(\frac{f_4}{f_1}\right) \tag{3.23}$$

Using (2.5) in (3.23), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(n)\,q^n = \left(\frac{f_6^2 f_{18}^4}{f_3^4 f_{36}^2} + q\frac{f_6^4 f_9^3 f_{36}}{f_3^5 f_{12} f_{18}^2} + 2q^2\frac{f_6^3 f_{18} f_{36}}{f_3^4 f_{12}}\right) \tag{3.24}$$

Extracting those terms that include powers of q with 1 $modulo$ 3 from (3.24) and replacing q with $q^{\frac{1}{3}}$, and thanks to Lemma. 2.6, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3n+1)\,q^n = \frac{f_2^4 f_3^3 f_{12}}{f_1^5 f_4 f_6^2} = \frac{f_2^4 f_{12}}{f_4 f_6^2}.\left(\frac{f_3^3}{f_1}\right).\left(\frac{1}{f_1^4}\right) \tag{3.25}$$

Employing (2.1) and (2.3) in (3.25),

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3n+1)\,q^n = \frac{f_4^6}{f_2^{12} f_8^4} + 4q\frac{f_4^4 f_8^4}{f_2^8} + \frac{f_4^{12} f_{12}^4}{f_2^{10} f_6^2 f_8^4} + 4q^2\frac{f_8^4 f_{12}^4}{f_2^6 f_6^2} \tag{3.26}$$

Extracting those terms that include q^{2n} and q^{2n+1} from (3.26) and replacing q with $q^{\frac{1}{2}}$, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+1)\,q^n = \frac{f_2^{16}}{f_1^{12} f_4^4} + 4q\frac{f_6^4 f_4^4}{f_1^6 f_3^2} \tag{3.27}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+4)\,q^n = 4\frac{f_2^4 f_4^4}{f_1^8} + \frac{f_6^4 f_2^{12}}{f_1^{10} f_3^2 f_4^4} \tag{3.28}$$

Using Lemma. 2.5 in (3.27), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+1)\,q^n \equiv \frac{f_4^8}{f_1^{12} f_4^4}\ (mod\ 4) \equiv f_1^4\ (mod\ 4)$$
$$= f_2^2\ (mod\ 4) \tag{3.29}$$

Extracting the terms containing q^{2n+1} from (3.29), we have

$$\overline{ped}_{3,6}(12n+7) \equiv 0\ (mod\ 4)$$

Extracting the terms containing q^{2n} from (3.29), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(12n+1)\,q^n \equiv f_1^2\ (\,mod\ 4) \tag{3.30}$$

Utilizing (2.2) in (3.30), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(12n+1)\,q^n \equiv \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right) (mod\ 4) \tag{3.31}$$

Extracting those terms containing q^{2n+1} from (3.31), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(24n+13)\,q^n \equiv 2\,\frac{f_1 f_8^2}{f_4}\ (mod\ 4) \equiv 2\, f_1\, \psi\,(q^4)\ (mod\ 4) \tag{3.32}$$

(3.32) is the $\alpha = 0$ case of (3.20). Assume (3.20) holds for some $\alpha \geq 0$. Using Lemma. 2.4 and Lemma. 2.5 in (3.20) and following the same steps as in the proof of (3.1), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.p^{2\delta+1}.n + 13.p^{2\delta+2}\right) q^n \equiv 2\, f_p\, \psi\,(q^{4p})\ (mod\ 4) \tag{3.33}$$

Extracting those terms that include q^{pn} from (3.33) and replacing q with $q^{\frac{1}{p}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.p^{2\delta+2}.n + 13.p^{2\delta+2}\right) q^n \equiv 2\, f_1\, \psi\,(q^4)\ (mod\ 4)$$

This shows (3.20) is valid for $\alpha + 1$. Thus, by induction (3.20) holds for all integer $n \geq 0$ and $\alpha \geq 0$.

Finally, extracting terms containing q^{pn+m}, where $1 \leq m \leq p-1$ from (3.33), we obtain (3.22).

Theorem. 4.2 *Suppose p is any prime such that* $\left(\frac{-9}{p}\right) = -1$ *and* $1 \leq u \leq p-1$. *For any integer* $n \geq 0$ *and* $\beta \geq 0$, *we have*

$$\overline{ped}_{3,6}(24n+22) \equiv 0 \ (mod\ 4) \tag{3.34}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(24.p^{2\alpha}.n` + 10.p^{2\alpha})q^n \equiv 2\ f_1\, \psi\,(q^3)\ \ (mod\ 4) \tag{3.35}$$

$$and \quad \overline{ped}_{3,6}(24.p^{2\alpha+1}.(pn` + u) + 10.p^{2\alpha+2})q^n \equiv 0 \ \ (mod\ 4) \tag{3.36}$$

Proof. Using Lemma. 2.6 in (3.28), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+4)\,q^n \equiv \frac{f_3^6 f_4^2}{f_1^{10}} \ \ (mod\ 4)$$

$$\equiv \frac{f_6^2 f_3^2}{f_1^2} (\ mod\ 4)$$

Utilizing (2.4) in (3.37), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+4)\,q^n \equiv f_6^2.\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right)$$

Extracting those terms that contains q^{2n+1}, and replacing q by $q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(12n+10)\,q^n \equiv 2\frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \ (mod\ 4)$$

Thanks to Lemma. 2.6, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(12n+10)\,q^n \equiv 2\ f_2\ f_6^3 (\ mod\ 4) \tag{3.38}$$

Extracting the terms containing q^{2n+1} from (3.38), we obtain (3.34).

Extracting terns containing q^{2n} from (3.38) and then replacing q with $q^{\frac{1}{2}}$, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(24n+10)\,q^n \equiv 2\ f_1\ f_3^3\ (\ mod\ 4)$$

(3.35) holds for $\alpha = 0$. Suppose (3.35) holds for some integer $\alpha \geq 0$. Utilizing Lemma 2.4 and Lemma. 2.5 in (3.20) in (3.35), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(24.p^{2\alpha}.n` + 10.p^{2\alpha})q^n$$

$$\equiv 2\left[\sum_{\substack{x=-\frac{p-1}{2} \\ x\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^x q^{\frac{3x^2+x}{2}} f\left(-q^{\frac{3p^2+(6x+1)p}{2}}, -q^{\frac{3p^2-(6x+1)p}{2}}\right)\right.$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\left|\left[\sum_{n=0}^{\infty} q^{\frac{3(y^2+y)}{2}} f(q^{\frac{3(p^2+(2y+1)p)}{2}}, q^{\frac{3(p^2-(2y+1)p)}{2}})\right.\right.$$

$$\left.+ q^{\frac{3(p^2-1)}{8}} \psi\left(q^{3p^2}\right)\right] (mod\ 4) \qquad (3.39)$$

Consider the congruence

$$\frac{3x^2+x}{2} + \frac{3(y^2+y)}{2} \equiv \frac{10(p^2-1)}{24}\ (mod\ p)$$

Above congruence implies

$$(6x+1)^2 + 9(2y+1)^2 \equiv 0\ (mod\ 4)$$

For $\left(\frac{-9}{p}\right) = -1$, the last congruence has only the solution $x = \frac{\pm p-1}{6}$ and $y = \frac{p-1}{2}$. Therefore collecting those terms that contains $q^{pn+\frac{10(p^2-1)}{24}}$ from (3.39), dividing by $q^{\frac{10(p^2-1)}{24}}$ and then replacing q with $q^{\frac{1}{p}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(24.p^{2\alpha+1}.n` + 10.p^{2\alpha+2})q^n \equiv 2\ f_p\ \psi(q^{3p})\ (mod\ 4) \qquad (3.40)$$

Extracting those terms that include q^{pn} from (3.40), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(24.p^{2\alpha+2}.n` + 10.p^{2\alpha+2})q^n \equiv 2\ f_1\ \psi(q^3)\ (mod\ 4)$$

Thus (3.35) holds for $\alpha + 1$. Thus (3.35) always holds.

Finally, extracting those terms containing $q^{pn+u}. 1 \leq u \leq p-1$ (3.40), we obtain 3.36).

Theorem 4.3. *For all integers* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$\overline{ped}_{3,6}\ (6n+5) \equiv 0\ (mod\ 4) \qquad (3.41)$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(3.2^{2\alpha+1}.n + 2^{2\alpha+1})q^n \equiv 2\frac{f_3^3}{f_1} \,(mod\ 4) \quad (3.42)$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(3.2^{2\alpha+3}.n + 5.2^{2\alpha+2})q^n \equiv 0 \,(mod\ 4) \quad (3.43)$$

Proof. From (3.24), utilizing Lemma 2.6 we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3n+2) = 2\frac{f_2^3 f_6 f_{12}}{f_1^4 f_4} \equiv 2\frac{f_6^3}{f_2} \,(mod\ 4) \quad (3.44)$$

Extracting the terms containing q^{2n} and q^{2n+1} from (3.42) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(6n+2)q^n \equiv 2\frac{f_3^3}{f_1} \,(mod\ 4) \quad (3.45)$$

$$and \quad \overline{ped}_{3,6}\,(6n+5) \equiv 0 \,(mod\ 4)$$

(3.45) is the $\alpha = 0$ case of (3.42). Assume (3.42) is valid for some integer $\alpha \geq 0$. Employing (2.23) in (3.42), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3.2^{2\alpha+1}.n + 2^{2\alpha+1})q^n \equiv 2\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)(mod\ 4) \quad (3.46)$$

Extracting those terms that include odd exponent of q from (3.46) , we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3.2^{2\alpha+2}.n + 2^{2\alpha+3})q^n \equiv 2\frac{f_6^3}{f_2} \,(mod\ 4) \quad (3.47)$$

Similarly, extracting those terms containing q^{2n} from (3.47), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3.2^{2\alpha+3}.n + 2^{2\alpha+3})q^n \equiv 2\frac{f_6^3}{f_2} \,(mod\ 4)$$

Above congruence is the $\alpha + 1$ case of (3.42). Thus (3.42) holds for all integer $n \geq 0$ and $\alpha \geq 0$.

Finally, extracting terms of the form q^{2n+1} from (3.47), we obtain (3.43).

Theorem. 4.4. *For all integer* $n \geq 0$ *and* $j \in \{1,2,3\}$, *we have*

$$\overline{ped}_{3,6}(3.2^{2\alpha+3}.n + 7.2^{2\alpha+1}) \equiv 0 \,(mod\ 4) \quad (3.48)$$

$$\overline{ped}_{3,6}(3.2^{2\alpha+4}.n + (6j+1).2^{2\alpha+1}) \equiv 0 \,(mod\ 4) \quad (3.49)$$

Proof. Extracting those terms of the type q^{2n} from (3.46 and then employing Lemma. 2.6, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3.\,2^{2\alpha+2}.\,n + 2^{2\alpha+1})q^n \equiv 2\frac{f_2^3 f_3^2}{f_1^2 f_6}\ (mod\ 4)$$
$$\equiv 2\ f_4\ (mod) \qquad (3.50)$$

Since R.H.S of (3.50) is a function of q^4, Therefore extracting the terms that include q^{2n+1} from (3.50), we arrive at (3.48).

Extracting those terms that include $q^{4n+j}, where\ j = 1, 2, 3$, we obtain (3.50).

Theorem.4.5. *Suppose* $p \geq 5$ *is any prime then for all integer* $n \geq 0, \alpha \geq 0\ , \beta \geq$ *and* $1 \leq j\ \leq p - 1$ *, we have*

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(3.\ 2^{2\alpha+4}.\,p^{2\beta}.\,n\ +\ 2^{2\alpha+1}.\ p^{2\beta}\right)q^n \equiv 2\ f_1\ (mod\ 4) \qquad (3.51)$$

$$\overline{ped}_{3,6}\left(3.\,2^{2\alpha+4}.\,p^{2\beta+1}(pn + j) + 2^{2\alpha+1}.\,p^{2\beta+2}\right) \equiv 0\ (mod\ 4) \qquad (3.52)$$

Proof. Extracting terms that include q^{4n} from (3.50) and replacing q by $q^{\frac{1}{4}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\,(3.\,2^{2\alpha+4}.\,n + 2^{2\alpha+1})q^n \equiv 2\ f_1\ (mod\ 4)$$

which is the $\beta = 0$ case of (3.51). Assume (3.51) holds for some integer $\beta \geq 0.$ Employing Lemma. 2.6, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\ \left(3.\,2^{2\alpha+4}.\,p^{2\beta}.\,n + 2^{2\alpha+1}.\ p^{2\beta}\right)q^n$$
$$\equiv 2\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\right] (mod\ 4) \qquad (3.53)$$

Extracting those terms containing $q^{pn+\frac{p^2-1}{24}}$ from (3.53) and then dividing both sides by $q^{\frac{p^2-1}{24}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(3.\,2^{2\alpha+4}.\,p^{2\beta+1}.\,n + 2^{2\alpha+1}.\ p^{2\beta+2}\right)q^n \equiv 2\ f_p(\ mod\ 4) \qquad (3.54)$$

Extracting those terms containing q^{pn} from (3.54), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(3.2^{2\alpha+4}.p^{2\beta+2}.n + 2^{2\alpha+1}.\ p^{2\beta+2}\right)q^n \ \equiv 2\, f_1(\,mod\ 4)$$

Thus, (3.51) holds for $\beta + 1$. By induction (3.51) always holds.

Extracting the terms containing $q^{pn+j}, 1 \le j\ \le p-1$ from (3.54), and replacing q^p by q, we obtain (3.52).

Theorem. 4.6 *Suppose p is any prime such that* $\left(\frac{-2}{p}\right) = -1$ *and* $1 \le v\ \le p-1$. *For any integer* $n \ge 0$ *and* $\mu \ge 0$, *we have*

$$\overline{ped}_{3,6}(24n+23) \equiv 0\ (mod\ 16) \tag{3.55}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(\,p^{2\mu}(24n+11)q^n \equiv 8\, f_2\ \ \psi(q^3)\ (mod\ 16) \tag{3.56}$$

$$\overline{ped}_{3,6}(24.\ p^{2\mu+1}.(pn+v) + 11.p^{2\mu+2})\ \equiv 0\ (mod\ 16) \tag{3.57}$$

Proof. Extracting those terms that include q^{3n+2} from (3.24), replacing q with $q^{\frac{1}{3}}$, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(3n+2)q^n = 2\frac{f_2^3 f_6 f_{12}}{f_1^4 f_4} = 2\frac{f_2^3 f_6 f_{12}}{f_4}.\left(\frac{1}{f_1^4}\right) \tag{3.58}$$

Employing (2.1) in (3.58) and extracting the odd term, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(6n+5)q^n \equiv\ \ 8\frac{f_3^3 f_4^4}{f_1^5}\ (mod\ 16)$$

$$\equiv 8\frac{f_3^3 f_4^3}{f_1}\ (mod\ 16) \tag{3.59}$$

Utilizing (2.3) in (3.59), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(6n+5)q^n \equiv\ 8\, f_4^3\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \tag{3.60}$$

Extracting the terms that include q^{2n+1} from (3.60), replacing q with $q^{\frac{1}{2}}$ we get

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(12n+11)q^n \equiv 8\, f_2^2\ f_6^3\ (mod\ 16) \tag{3.61}$$

Extracting the terms that include q^{2n+1} from (3.61), we obtain (3.55).

Extracting the terms that include q^{2n} from (3.61) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}(24n+11)q^n \equiv 8\, f_1^2\, f_3^3\ (mod\ 16)$$

$$\equiv 8\, f_2\, \psi(q^3)(\text{mod } 16)$$

Which shows that (3.55) is true for $\alpha = 0$. Assume (3.55) holds for some integer $\alpha \geq 0$. Employing Lemma 2.4 and Lemma. 2.5 in (3.55), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.\ p^{2\beta}.n + 11.p^{2\beta}\right)q^n$$

$$\equiv 8 \left[\sum_{\substack{x=-\frac{p-1}{2} \\ x \neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^x q^{3x^2+x} f\left(-q^{3p^2+(6x+1)p}, -q^{3p^2-(6x+1)p}\right) \right.$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{12}} f_{2p^2} \left[\left[\sum_{n=0}^{\infty} q^{\frac{3(y^2+y)}{2}} f(q^{\frac{3(p^2+(2y+1)p)}{2}}, q^{\frac{3(p^2-(2y+1)p)}{2}}) \right. \right.$$

$$\left. + q^{\frac{3(p^2-1)}{8}} \psi\left(q^{3p^2}\right) \right] (mod\ 16)$$

(3.62)

Consider the congruence

$$(3x^2+x) + \frac{3(y^2+y)}{2} \equiv \frac{11(p^2-1)}{24}\ (mod\ p)$$

which implies

$$2(6x+1)^2 + (6y+3)^2 \equiv 0\ (mod\ p) \tag{3.63}$$

For $\left(\frac{-2}{p}\right) = -1$, the congruence (3.63) has only the solution $x = \frac{\pm p-1}{6}$ and $y = \frac{p-1}{2}$. Therefore, extracting the terms that include $q^{pn+\frac{11(p^2-1)}{24}}$ from (3.63), dividing throughout by $q^{\frac{11(p^2-1)}{24}}$ and then replacing q^{pn} by q, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.\ p^{2\beta+1}.n + 11.p^{2\beta+2}\right)q^n \equiv 8\, f_{2p}\, \psi\,(q^{3p})\ (mod\ 16) \tag{3.64}$$

Extracting the terms that include q^{pn} from (3.64) and replacing q^p by q, we have

$$\sum_{n=0}^{\infty} \overline{ped}_{3,6}\left(24.\ p^{2\beta+2}.n + 11.p^{2\beta+2}\right)q^n \equiv 8\, f_2\, \psi\,(q^3)\,(mod\ 16)$$

which shows that (3.56) is true for $\beta + 1$. Hence, by induction (3.56) is proved.

Similarly, extracting the terms that include $q^{pn+v}, 1 \leq v \leq p-1$, we obtain (3.57).

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